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DYNAMICAL PROPERTIES OF EXPANSIVE ONE-SIDED CELLULAR AUTOMATA*

BY

F. BLANCHARD

Laboratoire de Mathématiques Discrètes
163 avenue de Luminy, Case 930, 13288 Marseille cedex 09, France
e-mail: blanchar@iml.univ-mrs.fr

AND

A. MAASS

Departamento de Ingeniería Matemática, Universidad de Chile
Casilla 170/3 correo 3, Santiago, Chile
e-mail: amaass@dim.uchile.cl

ABSTRACT

To every one-sided one-dimensional cellular automaton F with neighbourhood radius r we associate its **canonical factor** defined by considering only the first r coordinates of all the images of points under the powers of F. Whenever the cellular automaton is surjective, this factor is a subshift which plays a primary role in its dynamics. In this article we study the class of positively expansive one-sided cellular automata, i.e. those that are conjugate to their canonical factors. This class is a natural generalisation of the toggle or permutative cellular automata introduced in [He]. We prove that the canonical factors of positively expansive one-sided cellular automata are mixing subshifts of finite type that are shift equivalent to full shifts. Moreover, the uniform Bernoulli measure is the unique measure of maximal entropy for F. Consequently, their natural extensions are Bernoulli. We also describe a family of non-permutative positively expansive cellular automata.

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1. Introduction

Cellular automata (CA for short) were first used for modelling various physical and biological processes, and more recently also in computer science. The study of the mathematical properties of one-dimensional cellular automata started with the work of Hedlund [He]; various articles concerning different aspects of the dynamics of CA have been published since, for instance [Cov], [Cou], [G], [Hu], [L1], [M], [S], [W].

Surjective cellular automata are particularly attractive for ergodicians: in this case $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$, endowed with the continuous onto self-map F defining the automaton, is a topological flow, in other words a dynamical system "in equilibrium"; another nice feature is that the uniform measure is always invariant with respect to F. Among them, the class of permutative CA (or **toggle automata**), introduced by Hedlund in [He], has been thoroughly studied; its ergodic properties have been recently considered in [S]. The class of positively expansive cellular automata considered in this paper contains permutative automata on $A^{\mathbb{N}}$.

In this article we are only dealing with **one-sided** cellular automata (i.e. those acting on $A^{\mathbb{N}}$). To every surjective CA F: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ of radius r we associate its **canonical factor** S_F, generated by the first r coordinates of $F^n(x)$, $n \in \mathbb{N}$. One-sided permutative cellular automata exhibit the following dynamical properties: the canonical factor is a full shift and the flow $(A^{\mathbb{N}}, F)$ is conjugate to (S_F, σ). In this paper we deal with the symbolic dynamics of the family of CA that, as topological flows, are conjugate to one-sided subshifts — in other words, the abstract family of **positively expansive** one-sided cellular automata — and their ergodic behaviour when $A^{\mathbb{N}}$ is endowed with the uniform measure. Except in the Introduction, whenever we use the word expansive in this article we mean positively expansive.

It turns out that this expansiveness assumption has many strong symbolic and measure-theoretic consequences. Thus, on the symbolic side, every positively expansive one-sided CA is surjective, conjugate to a topologically mixing subshift of finite type that is shift equivalent to a full shift on k_1 letters.

Measure-theoretically, we use the previous results in order to identify the uniform measure λ over $A^{\mathbb{N}}$ to the Parry measure on (S_F, σ); thus λ has maximal entropy under the action of F — this is not true in general for surjective CA, as can be deduced from [Cov] and [L1] — and it is the unique measure with this property.

Finally, two examples, of completely different nature, allow us to show that the family of positively expansive cellular automata is substantially wider than that of toggle automata, for which Shereshevsky [S] first proved the same measure-theoretic results.

The article is laid out in four sections. Section 2 is devoted to definitions, examples and preliminary results. The main results are given in Section 3: we prove some combinatorial lemmas, from which one deduces the mixing property and the entropy of the CA, then show that the entropy of λ is the same, which proves that it is the Parry measure; finally this enables us to state an arithmetical condition for expansiveness and to prove the shift equivalence. We finally describe a family of positively expansive cellular automata which are not permutative even in a weaker sense; they are those representing the multiplication by k_1 in base p, when $p = k_1\, k_2$ and k_2 divides a power of k_1.

Finally here are a few remarks about positively expansive **two-sided** cellular automata (i.e., acting on $A^{\mathbb{Z}}$). Recently Nasu [N] proved a result that is strongly linked with ours: "any positively expansive endomorphism of a **two-sided** mixing subshift of finite type is conjugate to a one-sided full shift". This result describes completely the topological dynamics of this class of cellular automata, but there is no statement concerning the measure-theoretic dynamics of the system. On the other hand, if F is one-sided positively expansive, then its extension to $A^{\mathbb{Z}}$ is **never** positively expansive, which shows that the two questions are completely distinct. Nevertheless the underlying combinatorics are deeply similar; for instance, it is very easy to prove analogues of Propositions 2.2 to 2.4 for two-sided CA, not very hard to prove that of Theorem 3.8 (this is done in [Kû]), and we can also state that λ is the maximal measure for the dynamics. But the technicalities become increasingly different as one goes farther into the combinatorics of words, while the proofs do not really require new ideas. Also, the only positively expansive CA on $A^{\mathbb{Z}}$ we know are permutative in the strictest sense and were already studied in [He] and [S]. These are our reasons for choosing to treat the one-sided case only.

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many improvements.

2. Preliminaries

2.1 SYMBOLIC SYSTEMS. In this article we call (topological) **flow** a compact metric space X endowed with a surjective endomorphism T. Consider two flows (X,T) and (X',T'). We say that (X',T') is a factor of (X,T) if there exists a continuous onto map $\pi\colon X \to X'$ such that $T'\circ\pi = \pi\circ T$; if π is also 1-to-1 (X,T) and (X',T') are said to be conjugate. Let μ be a T-invariant measure. We endow (X',T') with the invariant measure μ' defined by $\mu'(A') = \pi\mu(A') = \mu(\pi^{-1}(A'))$ for any Borel set A' of X'.

A topological flow (X,T) is transitive if for any pair of open sets U, V of X there is a positive integer n such that $U \cap T^{-n}V \neq \emptyset$. Moreover, if there is $n_0 \in \mathbb{N}$ (only depending on U and V) such that for all $n \geq n_0$, $U \cap T^{-n}V$ is not empty, we say that the system is topologically mixing or simply **mixing**. The mixing and transitivity properties are conjugacy invariants.

In this article X is always a symbolic space. Let A be a finite alphabet. We denote by A^* the set of finite sequences or **words** on A, including the **empty word** 1; in other words, $A^* = \bigcup_{n\in\mathbb{N}} A^n$ where A^n is the set of words of length n of A^*; $|w|$ denotes the length of the word $w \in A^*$. A **language** L is an arbitrary subset of A^*; we denote by L_n the set of words of length n of L.

Let $K = \mathbb{N}$ or $\mathbb{Z}$. A^K is the set of infinite sequences $x = (x_i)_{i\in K}$, where $x_i \in A$; we call them **configurations**. For $i \leq j$ in K put $x(i,j) = x_i \cdots x_j$. Let x be a configuration and $w = w_0 \cdots w_{n-1}$ be a word of length n; we denote by wx the configuration y defined by $y(0, n-1) = w$ and $y_{n+i} = x_i$, $i \in \mathbb{N}$. A^K is endowed with the product topology and the **shift** $\sigma\colon A^K \to A^K$, $\sigma(x) = (x_{i+1})_{i\in K}$. When $K = \mathbb{Z}$, σ is a homeomorphism. The family of cylinder sets $[w]_i = \{x \in A^K\colon x(i, i+|w|-1) = w\}$, where $w \in A^*$ and $i \in K$, is a fundamental base of clopen neighbourhoods of A^K. For this topology A^K is a compact metric space; the distance is defined by

$$d(x,y) = \sum_{i\in K} \frac{d_i(x,y)}{2^{|i|+1}}, \quad \text{where } d_i(x,y) = 1 \text{ if } x_i \neq y_i \text{ and } 0 \text{ otherwise;}$$

two configurations are close to each other with respect to this distance if for some large n their coordinates coincide from $-n$ to $+n$. The flow (A^K, σ) is called **full shift** (one-sided full shift when $K = \mathbb{N}$, or simply full shift when $K = \mathbb{Z}$).

A **subshift** or **symbolic flow** is a closed shift-invariant subset S of A^K endowed with σ. In general we identify the flow (S, σ) with the space S. The language associated to the subshift S is

$$L(S) = \{w \in A^*: \exists\, x \in S, i \in K, \text{ such that } x(i, i + |w| - 1) = w\}.$$

It is well known that S, whether one- or two-sided, is completely described by its language, and a one-sided subshift is the projection of a two-sided one.

In this article **subshifts of finite type** (SFT) play a prominent role. A subshift $S \subseteq A^K$ is said to be of finite type if there exist a positive integer N (assumed to be minimal), and a collection L of words of length N, such that $x \in S$ if and only if $x(i, i + N - 1) \in L$ for all $i \in K$. The integer N is called the **order** of S. Any SFT S of order N is conjugate to a SFT of order 2 (called Markov system), by the map which associates to each $x \in S$ the point $y = (x(i, i + N - 2))_{i \in K} \in L_{N-1}(S)^K$. A Markov system, $S \subseteq A^K$, is associated to an incidence matrix $M(S)$ indexed by A such that the entry corresponding to $(a, b) \in A^2$ is equal to 1 if $ab \in L_2(S)$, 0 otherwise. The mixing properties of the flow (S, σ) are deduced from properties of its transition matrix. Thus, S is transitive if for all $(a, b) \in A^2$ there exists $n \in \mathbb{N}$ such that $M(S)^n(a, b) > 0$, and S is mixing if n can be chosen independently of the entry (a, b).

In the category of measurable dynamical systems, where topological structures are replaced by measurable ones, the natural notion of equivalence between systems is **isomorphism**, that is, existence of a bimeasurable bijection defined over a set of measure one, exchanging the measures and transformations.

2.2 ENTROPY. Let (X, T) be a topological flow and μ be a T-invariant measure. First, let us recall the notions of topological and measure-theoretic entropy.

For defining topological entropy one must introduce some additional definitions. Let $\mathcal{R}$ be an open cover of X: we denote by $H(\mathcal{R})$ the real number $\inf\{\log \operatorname{card}(\mathcal{R}')\}$, where the inf is taken over the set of finite subcovers $\mathcal{R}'$ of $\mathcal{R}$. Let $\mathcal{S}$ be another subcover of X; we say that $\mathcal{R}$ is **finer** than $\mathcal{S}$, and denote this by $\mathcal{S} \leq \mathcal{R}$, if for all $U \in \mathcal{R}$ there exists $V \in \mathcal{S}$ such that $U \subset V$. This implies $H(\mathcal{S}) \leq H(\mathcal{R})$.

Denote by $\mathcal{R} \vee \mathcal{S}$ the cover made up of all the intersections $R \cap S$, where $R \in \mathcal{R}$ and $S \in \mathcal{S}$. The **topological entropy of the cover** $\mathcal{R}$ is the (well defined) nonnegative number

$$h(\mathcal{R},T)=\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}T^{-i}\mathcal{R}\right).$$

Whenever $\mathcal{S}\le\mathcal{R}$ one has $h(\mathcal{S},T)\le h(\mathcal{R},T)$.

By definition the **topological entropy** of a flow (X,T) is

$$h_{\text{top}}(X,T)=\sup h(\mathcal{R},T),$$

where the sup is taken over all finite open covers of X.

Here is a list of classical properties of topological entropy we shall use in the sequel; for more details see [DGS].

(1) Suppose that $(\mathcal{R}_n)_{n\in\mathbb{N}}$ is a generator, i.e. an increasing family of open covers of X with the property that for any other open cover $\mathcal{R}$ of X there is $n\in\mathbb{N}$ such that $\mathcal{R}\le\mathcal{R}_n$. Then,

$$h_{\text{top}}(X,T)=\lim_{n\to\infty}h(\mathcal{R}_n,T).$$

A consequence is that for a subshift S one has

$$h_{\text{top}}(S)=\lim_{n\to\infty}\frac{1}{n}\log L_n(S).$$

(2) If (X',T') is a factor of (X,T) then $h_{\text{top}}(X',T')\le h_{\text{top}}(X,T)$.

(3) Let $S\subseteq A^K$ be a Markov system; then $h_{\text{top}}(S)=\log\xi$, where ξ is the maximal eigenvalue of its transition matrix.

We shall only use the definition of measure-theoretic entropy for measures on subshifts. Consider a subshift $X\subseteq A^K$, where $K=\mathbb{N}$ or $\mathbb{Z}$. The **entropy of (X,T) with respect to the invariant measure μ** is

$$h_\mu(X,T)=-\lim_{n\to\infty}\frac{1}{n}\sum_{w\in L_n(X)}\mu([w]_0)\,\log\mu([w]_0). \tag{2.1}$$

In the general case, if the flows (X,T) and (X',T') are conjugate by the map $\pi\colon X\to X'$, then $h_\mu(X,T)=h_{\pi\mu}(X',T')$.

Both notions of entropy, topological and measure-theoretic, are linked by the classical "variational principle"

$$h_{\text{top}}(X,T)=\sup h_\mu(X,T),$$

where the sup is taken over the set of invariant measures of the flow (X,T). A measure μ for which $h_\mu(X,T)=h_{\text{top}}(X,T)$ is said to be a **measure of maximal entropy**.

2.3 CELLULAR AUTOMATA. A one-sided **cellular automaton** (CA for short) is a map $F: A^{\mathbb{N}} \to A^{\mathbb{N}}$ defined for $x = (x_i)_{i\in\mathbb{N}} \in A^{\mathbb{N}}$, $i \in \mathbb{N}$ by

$$F(x)_i = f(x_i x_{i+1} \cdots x_{i+r}),$$

where $f: A^{r+1} \to A$ is a given **local map** or rule. The integer r is called the (neighbourhood) **radius** of the CA. F is continuous and commutes with the shift σ. The local rule f can also be applied to the words of length $m \geq r+1$:

$$f(a_1, \ldots, a_m) = f(a_1, \ldots, a_{r+1}) \cdots f(a_{m-r}, \ldots, a_m) \quad \text{for all } (a_1, \ldots, a_m) \in A^m.$$

For w, $w' \in A^* \cup A^{\mathbb{N}}$, we say that w' is a **successor** of w with respect to f, and denote this by $w \xrightarrow{f} w'$, if there are $x_w = wy_w$ and $x_{w'} = w'y_{w'}$ in $A^{\mathbb{N}}$ such that $F(x_w) = x_{w'}$.

The following proposition, easily deduced from a result of Hedlund ([He,theorem 5.4]), is a very useful combinatorial characterization of onto CA.

PROPOSITION 2.1: *Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be a CA. The following conditions are equivalent:*

(i) *F is onto.*

(ii) *For any $w \in A^*$,* $\mathrm{card}(\{w' \in A^{|w|+r}: f(w') = w\}) = \mathrm{card}(A)^r$.

A direct consequence is that the uniform Bernoulli measure λ of $A^{\mathbb{N}}$ defined by

$$\lambda([w]_i) = \frac{1}{\mathrm{card}(A)^n}, \quad \text{for all } w \in A^n, \quad i \in \mathbb{N},$$

is invariant with respect to F whenever the cellular automaton is surjective.

Let $F: A^{\mathbb{N}} \to A^{\mathbb{N}}$ be an onto CA of radius r. We associate to F, in a natural way, its **canonical subshift**

$$S_F = \{y \in (A^r)^{\mathbb{N}}:\ \exists\, x \in A^{\mathbb{N}}, \text{ such that } y_i = F^i(x)(0, r-1),\ i \in \mathbb{N}\}.$$

It is clear that S_F is closed and shift-invariant. The flow (S_F, σ) is a symbolic factor of $(A^{\mathbb{N}}, F)$ through the factor map $\pi_F : A^{\mathbb{N}} \to S_F$ defined by $\pi_F(x)_i = F^i(x)(0, r-1)$ for all $x \in A^{\mathbb{N}}$, $i \in \mathbb{N}$. In other words π_F associates to each point of $A^{\mathbb{N}}$ its itinerary with respect to the partition $\mathcal{P} = \{[w]_0: w \in A^r\}$. This application is in general not bijective and the language $L(S_F)$ is context-sensitive (see, for example, [G]). In the sequel we shall endow S_F with the invariant measure $\lambda_F = \pi_F(\lambda)$.

The subshift S_F is developing into an important tool for the study of one-sided cellular automata F (cf. [Cov]); it is easy to show that its entropy is equal to that of F.

2.4 POSITIVELY EXPANSIVE CELLULAR AUTOMATA. An endomorphism T of a compact metric space X is **positively expansive** if there is a constant $\epsilon > 0$ such that for all x, $y \in X$, $x \neq y$, there exists $n \in \mathbb{N}$ for which $d(T^n(x), T^n(y)) > \epsilon$, where $d(\cdot,\cdot)$ denotes the distance in X. Expansiveness is classically defined for homeomorphisms, and the formal definition is the same except that $n \in \mathbb{Z}$. Since CA maps are generally not one-to-one, the natural notion in the present setting is that of positive expansiveness. In the sequel, whenever we use the word expansive this means positively expansive.

Let us first describe some examples of positively expansive CA.

Example 1: Toggle automata: A cellular automaton F: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a toggle automaton if for every $w \in A^r$ there is a permutation ρ_w of A such that $f(wa_r) = \rho_w(a_r)$ for all $a_r \in A$.

If F is a toggle automaton it is easy to check that for all w, $w' \in A^r$ there exists a unique $w'' \in A^r$ such that $ww'' \xrightarrow{f} w'$. Moreover, this property implies that F is positively expansive and that the canonical factor S_F is equal to $(A^r)^{\mathbb{N}}$.

Example 2: A generalisation: We construct a family of automata that are permutative in a weaker sense: a permutation is attached to every word in a finite set $\mathcal{C}$, but here the set is no longer A^r. Fix an integer $r > 2$ and consider a finite language $\mathcal{C} \subseteq A^r \cup A^{r-1}$ such that:

(1) $\mathcal{C}$ is a complete prefix code: if $c_0 \cdots c_k \in \mathcal{C}$, then for all $0 \leq i < k$, $c_0 \cdots c_i \notin \mathcal{C}$, and if $x \in A^{\mathbb{N}}$, there exist (unique) $c \in \mathcal{C}$ and $y \in A^{\mathbb{N}}$ such that $x = cy$.

(2) If $c_0 \cdots c_{r-1} \in \mathcal{C}$, then for all $\alpha \in A$, $c_1 \cdots c_{r-2}\alpha \in \mathcal{C}$.

For instance, for $r = 3$ and $A = \{0, 1\}$, $\mathcal{C} = \{00, 01, 11, 100, 101\}$ satisfies these two conditions.

Now, we associate to each word $w \in \mathcal{C}$ a permutation ρ_w: $A \to A$ with the following property: let $w = a_0a_1 \cdots a_{r-2}\alpha \in \mathcal{C}$, $w' = a_0a_1 \cdots a_{r-2}\alpha' \in \mathcal{C}$, where $\alpha \neq \alpha'$; if for b, $b' \in A$, $\rho_w(b) = \rho_{w'}(b')$ then $\rho_{(a_1 \cdots a_{r-2}\alpha)}(b) \neq \rho_{(a_1 \cdots a_{r-2}\alpha')}(b')$. This choice is possible: first fix arbitrarily the permutations associated with the words of length $r - 1$, and then choose the missing ones in such a way that the property holds.

Finally, define the cellular automaton F_C: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ such that for all $x \in A^{\mathbb{N}}$ and $i \in \mathbb{N}$, $F_C(x)_i = \rho_w(x_{i+|w|})$, where w is the unique prefix of $x(i, i+r)$ in C. Evidently when $C \cap A^r = \emptyset$ we have defined a toggle automaton.

For F_C the following holds: for all w, $w' \in A^{r-1}$ there is a unique $a \in A$ such that $wa \xrightarrow{f_C} w'$. It follows that F_C is positively expansive and that S_{F_C} is conjugate to $((A^{r-1})^{\mathbb{N}}, \sigma)$.

It seems likely that one might define a larger, more natural class of "permutation automata", retaining the expansiveness property.

Here are some elementary properties of positively expansive CA.

PROPOSITION 2.2: *Let F be a positively expansive CA on $A^{\mathbb{N}}$; then F is onto.*

Proof: Let $k = \text{card}(A)$. $(F(A^{\mathbb{N}}), \sigma)$ is a factor of $(A^{\mathbb{N}}, \sigma)$; if $x \in F(A^{\mathbb{N}})$ and $F(y) = x$, since F is expansive there exists $r' \in \mathbb{N}$ such that $(F^i(y)(0, r'-1))_{i \in \mathbb{N}}$ determines y, consequently x has a bounded number of preimages by F. Therefore $h(A^{\mathbb{N}}, \sigma) = h(F(A^{\mathbb{N}}), \sigma) = \log k$; but the only subshift of $A^{\mathbb{N}}$ having this entropy is $A^{\mathbb{N}}$ itself; hence $F(A^{\mathbb{N}}) = A^{\mathbb{N}}$. ∎

PROPOSITION 2.3: *Let F: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ be a CA. The two following conditions are equivalent:*

(i) *F is positively expansive;*

(ii) *$(A^{\mathbb{N}}, F)$ and (S_F, σ) are topologically conjugate.*

Proof: Assume F is expansive with expansiveness constant ϵ, and take $r(\epsilon) \in \mathbb{N}$ such that $d(x, y) > \epsilon$ if and only if $x(0, r(\epsilon)-1) \neq y(0, r(\epsilon)-1)$, for x, $y \in A^{\mathbb{N}}$. We claim one can choose $r(\epsilon) \leq r$, where r is the radius of F: if this is false, there are two points $x \neq y \in A^{\mathbb{N}}$ such that $\pi_F(x) = \pi_F(y)$ (even if for some $n \in \mathbb{N}$, $F^n(x)(r, r(\epsilon)-1) \neq F^n(y)(r, r(\epsilon)-1)$). Fix $w \in A^{r(\epsilon)}$ and consider the points $x_w = wx$ and $y_w = wy$: for all $i \in \mathbb{N}$, $F^i(x_w)(0, r(\epsilon)-1) = F^i(y_w)(0, r(\epsilon)-1)$, which is a contradiction. Thus if F is expansive $\pi_F(x) \neq \pi_F(y)$ for $x \neq y$ in $A^{\mathbb{N}}$, and π_F is a conjugacy map. The sufficient condition is straightforward. ∎

The following lemma reminds one of the characterizations of shift-commuting or right resolving maps in Symbolic Dynamics. It relies on a simple compactness argument.

LEMMA 2.4: *Let F: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ be a positively expansive CA. There are a positive integer ℓ, and a map $\bar{f}$: $L_{\ell+1}(S_F) \to A^r$ such that for all $w = w_0 \cdots w_\ell \in$*

$L_{\ell+1}(S_F)$ and $x \in A^{\mathbb{N}}$ such that $F^i(x)(0, r-1) = w_i$, $i = 0, \ldots, \ell$, one has $x(r, 2r-1) = \bar{f}(w)$.

Remark 2.5: Given a finite subset $I \in \mathbb{N}^2$, by suitably many iterations of Lemma 2.4 one obtains that there is $n \in \mathbb{N}$ such that $F^i(x)(0, r-1)$, $i = 0, \ldots, n$ completely determine the set of coordinates $(F^j(x))_k$, $(j,k) \in I$.

Lemma 2.4 expresses the existence of the **transposed flow** $(S_F, \bar{F})$ of a positively expansive CA F, where $\bar{F} = \pi_F \circ \sigma^r \circ \pi_F^{-1}$ is also surjective; its local rule $\bar{f}$ is defined as follows: if y, $y' \in S_F$ and $\bar{F}(y) = y'$, then $y'_i = \bar{f}(y(i, i+\ell))$ for any $i \in \mathbb{N}$. Obviously $(S_F, \bar{F})$ is also positively expansive.

3. Results

In this section F is always a one-sided CA with radius r. An important case is when $r = 1$. Notice that F is always conjugate to a CA F': $(A^r)^{\mathbb{N}} \to (A^r)^{\mathbb{N}}$ with radius equal to 1. Indeed, define the local rule $f' : (A^r)^2 \to A^r$ associated to F' by $f'(w_1, w_2) = f(w_1, w_2)$ for w_1, $w_2 \in A^r$. The map $\phi : A^{\mathbb{N}} \to (A^r)^{\mathbb{N}}$ such that $\phi(x)_i = x(ir, ir+r-1)$ for every $x \in A^{\mathbb{N}}$ and $i \in \mathbb{N}$ defines a conjugacy between $(A^{\mathbb{N}}, F)$ and $((A^r)^{\mathbb{N}}, F')$. Naturally if F is positively expansive the conjugate map F' also is.

The class of positively expansive one-sided CA with radius 1 will be called (E1).

3.1 Topological properties. A cellular automaton is said to be **right-closing** if there is $n \in \mathbb{N}$ such that if $a_0 \xrightarrow{f} b_0 \cdots b_{n-1}$, $a_0 \in A^r$, $b_0 \cdots b_{n-1} \in A^{rn}$, then there is a unique $a_1 \in A^r$ such that $x(0, r-1) = a_0$ and $F(x)(0, rn-1) = b_0 \cdots b_{n-1}$ imply that $x(r, 2r+1) = a_1$.

Lemma 3.1: *Positively expansive CA are right-closing.*

Proof: Choose $n = \ell$ as given by Lemma 2.4 and take $a_0 \in A^r$, $b_0 \cdots b_{n-1} \in A^{rn}$ such that $a_0 \xrightarrow{f} b_0 \cdots b_{n-1}$. The condition $F(x)(0, rn-1) = b_0 \cdots b_{n-1}$ completely determines $F^i(x)(0, r-1)$, $i = 1 \cdots n$; together with $x(0, r-1) = a_0$, by Lemma 2.4 this determines $x(r, 2r+1)$. ∎

The next two results are essentially in [Ků]. The first describes a property of right-closing CA (which are not necessarily expansive). The claim at the beginning of the proof was first proved by Kitchens [Ki].

PROPOSITION 3.2: *Right-closing CA are open.*

Proof: By recoding assume F is a right-closing CA with radius 1 and let n be as in the definition of right-closing maps. We prove the following property:

$a_0 \in A$ and $b_0 \cdots b_{n-1} \in A^n$ are such that $a_0 \xrightarrow{f} b_0 \cdots b_{n-1}$. Then for all $\beta \in A$ there is a unique $\alpha \in A$ such that $a_0\alpha \xrightarrow{f} b_0 \cdots b_{n-1}\beta$.

Suppose that the claim is not true, then there exist $a_0, \beta \in A$ and $b_0 \cdots b_{n-1} \in A^n$ with $a_0 \xrightarrow{f} b_0 \cdots b_{n-1}$ but $b_0 \cdots b_{n-1}\beta$ is not a successor of a_0. Put $A = \{a_0, a_1, \ldots, a_{|A|-1}\}$.

Since F is onto there is $\alpha \in A$ such that the word $b_0 \cdots b_{n-1}\beta$ is one of its successors. Without loss of generality assume that $a_1 \xrightarrow{f} b_0 \cdots b_{n-1}\beta$ and take $v_0 \cdots v_n \in A^{n+1}$ such that $f(a_1 v_0 \cdots v_n) = b_0 \cdots b_{n-1}\beta$. Obviously

$$a_1 v_0 \cdots v_n a_0 \xrightarrow{f} b_0 \cdots b_{n-1}\beta\delta b_0 \cdots b_{n-1}$$

for some $\delta \in A$, but $w_1 = b_0 \cdots b_{n-1}\beta\delta b_0 \cdots b_{n-1}\beta$ is not a successor of a_1 because by right-closingness any preimage of w_1 must have the suffix $a_1 v_0 \cdots v_n a_0$. Therefore, w_1 is a successor neither of a_0 nor of a_1.

We can repeat the same construction to extend the word w_1 to the right until one obtains a word $w \in A^*$ without a preimage in A^*, which is a contradiction. This proves the claim.

Thus the image of any cylinder set is a finite union of cylinder sets; therefore F is an open map. ∎

THEOREM 3.3: *Let F be a positively expansive CA. Then it is conjugate to a subshift of finite type.*

Proof: By Proposition 3.2, F is open. This implies that the conjugate map $\sigma\colon S_F \to S_F$ is also open. Parry's result that one-sided subshifts for which the shift is an open map are exactly the SFT [P2] ends the proof. ∎

Suppose $N > \ell$ is an order of the subshift of finite type S_F, where ℓ is the constant from Lemma 2.4, which is also equal to the right-closing constant.

LEMMA 3.4: *Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be a CA of type* (E1). *Suppose that $a_0 \cdots a_{N-1} \in A^N$ and $b_1 \cdots b_{N-1} \in A^{N-1}$ are such that $a_0 \xrightarrow{f} b_1 \cdots b_{N-1}$ and for all $i \in \{0, \ldots, N-2\}$, $f^i(b_1 \cdots b_{N-1})(0) = a_{i+1}$. Then for all $b'_1 \cdots b'_{N-1} \in A^{N-1}$ such that $f^i(b'_1 \cdots b'_{N-1})(0) = a_{i+1}$, $i \in \{0, \ldots, N-2\}$, one has $a_0 \xrightarrow{f} b'_1 \cdots b'_{N-1}$.*

Proof: By hypothesis $a_0 \cdots a_{N-1} \in L_N(S_F)$. Let $b'_1 \cdots b'_{N-1} \in A^{N-1}$ be such that $f^i(b'_1 \cdots b'_{N-1})(0) = a_{i+1}$, $i \in \{0, \ldots, N-2\}$, and consider the configuration

$z' = b'_1 \cdots b'_{N-1} z'_0 z'_1 \cdots \in A^{\mathbb{N}}$, where $(z'_i)_{i \in \mathbb{N}}$ is arbitrary. Since $\pi_F(z')_i = a_{i+1}$ for all $i \in \{0, \ldots, N-2\}$ and the order of S_F is N, then $a_0 \pi_F(z') \in S_F$. It follows that $F(a_0 x') = z'$ for at least one $x' \in A^{\mathbb{N}}$. This completes the proof. ∎

Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be positively expansive. For $a_1 \cdots a_{N-1} \in L_{N-1}(S_F)$ define

$$k_1(a_1 \cdots a_{N-1}) = \mathrm{card}(\{\, a_0 \in A^r \colon a_0 a_1 \cdots a_{N-1} \in L_N(S_F) \,\}). \tag{3.1}$$

By Lemma 3.4, using the conjugacy of F with a CA of type (E1) described at the beginning of this section, one gets

$$k_1(a_1 \cdots a_{N-1}) = \mathrm{card}(\{\, a_0 \in A^r \colon a_0 \xrightarrow{f} b_1 \cdots b_{N-1} \,\}) \tag{3.2}$$

for any choice of the words $b_1, \ldots, b_{N-1} \in A^r$ such that $f^i(b_1 \cdots b_{N-1})(0, r-1) = a_{i+1}$, $i \in \{0, \ldots, N-2\}$. On the other hand, for $a_0 \in A^r$ and $b_1, \ldots, b_{N-1} \in A^r$ such that $a_0 \xrightarrow{f} b_1 \cdots b_{N-1}$ define

$$k_2(a_0 \,; b_1 \cdots b_{N-1}) = \mathrm{card}(\{\, w \in A^{r(N-1)} \colon f(a_0 w) = b_1 \cdots b_{N-1} \,\}). \tag{3.3}$$

LEMMA 3.5: *Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be positively expansive. The map $k_2(\cdot \,;\, \cdot)$ is constant over the pairs of words $(a_0 \,; b_1 \cdots b_{N-1}) \in A^r \times (A^r)^{N-1}$ such that $a_0 \xrightarrow{f} b_1 \cdots b_{N-1}$.*

Proof: Since the statement is about the combinatorics of words with length a multiple of r, assume F is of type (E1).

Put provisionally $c = \min(k_2(a_0; b_1 \cdots b_{N-1}))$, where $(a_0; b_1 \cdots b_{N-1}) \in A \times A^{N-1}$ and $a_0 \xrightarrow{f} b_1 \cdots b_{N-1}$. Let $(\bar{a}_0; \bar{b}_1 \cdots \bar{b}_{N-1})$ be such that $k_2(\bar{a}_0; \bar{b}_1 \cdots \bar{b}_{N-1}) = c$. Then there exist exactly c different words $w_1, \ldots, w_c \in A^{N-1}$ such that $f(\bar{a}_0 w_i) = \bar{b}_1 \cdots \bar{b}_{N-1}$ for all $i \in \{1, \ldots, c\}$; let α_i and γ_i be the first and last letters of w_i.

By Proposition 3.2, for every $\beta \in A$ there is a unique $\alpha(\beta) \in \{\alpha_1, \ldots, \alpha_c\}$ such that $\bar{a}_0 \alpha(\beta) \xrightarrow{f} \bar{b}_1 \cdots \bar{b}_{N-1} \beta$. Moreover, if $v \in A^{N-1}$ is such that $f(\alpha(\beta) v) = \bar{b}_2 \cdots \bar{b}_{N-1} \beta$ then one of the words w_i, $i \in \{1, \ldots, c\}$ must be a prefix of $\alpha(\beta) v$. There are exactly c card(A) words of length N having each one a prefix in the family $\{w_1, \ldots, w_c\}$. Therefore, we conclude that

$$c\,\mathrm{card}(A) \geq \sum_{\beta \in A} k_2(\alpha(\beta); \bar{b}_2 \cdots \bar{b}_{N-1} \beta).$$

Since c is minimal, it follows that for all $\beta \in A$, $k_2(\alpha(\beta); \bar{b}_2 \cdots \bar{b}_{N-1}\beta) = c$.

We have proved that if $\bar{a}_0 \xrightarrow{f} \bar{b}_1 \cdots \bar{b}_{N-1}$ and $k_2(\bar{a}_0 ; \bar{b}_1 \cdots \bar{b}_{N-1}) = c$, then for any α, $\beta \in A$ such that $\bar{a}_0\alpha \xrightarrow{f} \bar{b}_1 \cdots \bar{b}_{N-1}\beta$ one has $k_2(\alpha ; \bar{b}_2 \cdots \bar{b}_{N-1}\beta) = c$.

Now, let $(a_0 ; b_1 \cdots b_{N-1}) \in A \times A^{N-1}$ be such that $a_0 \xrightarrow{f} b_1 \cdots b_{N-1}$ and consider $\beta_1 \in A$ such that $f(\gamma_1 a_0) = \beta_1$. Then, $\bar{a}_0 w_1 \xrightarrow{f} \bar{b}_1 \cdots \bar{b}_{N-1}\beta_1 b_1 \cdots b_{N-2}$ and $w_1 a_0$ is the unique word of A^N which satisfies $\bar{a}_0 w_1 a_0 \xrightarrow{f} \bar{b}_1 \cdots \bar{b}_{N-1}\beta_1 b_1 \cdots b_{N-1}$. Applying this last result N times in order to reach the pair $(a_0 ; b_1 \cdots b_{N-1})$ we deduce that $k_2(a_0 ; b_1 \cdots b_{N-1}) = c$, which implies the result. ∎

The constant defined above will be denoted by k_2.

PROPOSITION 3.6: *Let F: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ be positively expansive. Then $k_1(\cdot)$: $L_{N-1}(S_F) \to \mathbb{N}$ is a constant denoted by k_1 and* $\operatorname{card}(A)^r = k_1\ k_2$.

Proof: Keep the notations from the last lemma. Let $b_1, \ldots, b_{N-1} \in A^r$. Define $a_{i+1} = f^i(b_1 \cdots b_{N-1})(0, r-1)$ for $i \in \{0, \ldots, N-2\}$. By Proposition 2.1, $\operatorname{card}(\{\ w \in A^{rN}: f(w) = b_1 \cdots b_{N-1}\ \}) = \operatorname{card}(A)^r$, hence

$$\begin{aligned}\operatorname{card}(A)^r &= \sum_{a_0 \in A^r\ : a_0 \xrightarrow{f} b_1 \cdots b_{N-1}} k_2(a_0 ; b_1 \cdots b_{N-1}) \\ &= k_2\ \operatorname{card}(\{\ a_0 \in A^r: a_0 \xrightarrow{f} b_1 \cdots b_{N-1}\ \}).\end{aligned}$$

Finally, from equality (3.2), we conclude that

$$k_1(a_1 \cdots a_{N-1}) = \operatorname{card}(\{\ a_0 \in A: a_0 \xrightarrow{f} b_1 \cdots b_{N-1}\ \}) = \frac{\operatorname{card}(A)^r}{k_2}. \quad ∎$$

In Example 1 the value of k_1 is $\operatorname{card}(A)^r$ and in Example 2 it is $\operatorname{card}(A)^{r-1}$. We shall see in Section 4 that for any $k_1, k_2 \in \mathbb{N}$ such that k_2 divides an integer power of k_1 we can construct a CA of type (E1) where the cardinality of the alphabet is equal to $k_1 \cdot k_2$.

Here are the two first topological consequences of the results above:

COROLLARY 3.7: *Let F: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ be positively expansive. Then* $h_{\text{top}}(A^{\mathbb{N}}, F) = h_{\text{top}}(S_F) = \log k_1$.

Proof: This follows immediately from Propositions 2.3 and 3.6. ∎

THEOREM 3.8: *Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be positively expansive. Then F is topologically mixing and S_F is a mixing SFT.*

Proof: By recoding we can suppose that F is of type (E1). Since by Proposition 2.3, $(A^{\mathbb{N}}, F)$ is conjugate to (S_F, σ), it is enough to prove that S_F is mixing.

Theorem 3.3 tells us S_F is a SFT; recall N is the order of S_F. To show S_F is mixing it is enough to find a constant $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$ and $w = w_0 \cdots w_{N-2}$, $\tilde{w} = \tilde{w}_0 \cdots \tilde{w}_{N-2} \in L_{N-1}(S_F)$, there exists $u \in A^t$ satisfying $wu\tilde{w} \in L(S_F)$.

First observe that there is a constant m_0, that we choose larger than $3N$, such that if $x, y \in A^{\mathbb{N}}$ satisfy $F^i(x)(0) = F^i(y)(0)$, $i = 0, \ldots, m_0 - 1$, then

$$F^0(x)(N-1) \cdots F^{N-2}(x)(N-1) = F^0(y)(N-1) \cdots F^{N-2}(y)(N-1);$$

this is proved by applying Lemma 2.4 repeatedly. In other words this means that the first m_0 coordinates of $\pi_F(x)$ completely determine the first $N-1$ coordinates of $\pi_F(\sigma^{N-1}x)$.

Fix a point $x \in A^{\mathbb{N}}$ in such a way that w is a prefix of $(F^i(x)(N-1))_{i \in \mathbb{N}}$, that is, $w = F^0(x)(N-1) \cdots F^{N-2}(x)(N-1)$. One can do this because $w \in L(S_F)$. Put $v = F^0(x)(0) \cdots F^{m_0-1}(x)(0) = \tilde{v}\tilde{w}'$, with $|\tilde{w}'| = N-1$. Of course $v \in L(S_F)$ too. Now we forget about x and only keep v in mind.

Now consider another point $\tilde{x} \in A^{\mathbb{N}}$ such that $\tilde{w}' = F^0(\tilde{x})(0) \cdots F^{N-2}(\tilde{x})(0)$ and $\tilde{w} = F^0(\tilde{x})(N-1) \cdots F^{N-2}(\tilde{x})(N-1)$. To construct it, since $\tilde{w}$ and $\tilde{w}'$ belong to $L_{N-1}(S_F)$, we only have to concatenate a preimage of $\tilde{w}'$ of length $N-1$ with a preimage of length $N-1$ of $\tilde{w}$, then arbitrarily complete this word to the right into a configuration. Let $\pi_F(\tilde{x})$ be the unique element of S_F associated to $\tilde{x}$.

Since the order of S_F is N, one has $\tilde{v}\pi_F(\tilde{x}) \in S_F$. Since F is expansive there is $y \in A^{\mathbb{N}}$ such that $\pi_F(y) = \tilde{v}\pi_F(\tilde{x})$ and $F^{m_0-N+1}(y) = \tilde{x}$. But since v is a prefix of length m_0 of $\pi_F(y)$, then by the observation above w is a prefix of $\pi_F(\sigma^{N-1}y)$; therefore $F^0(y)(N-1) \cdots F^{m_0-1}(y)(N-1) = wu\tilde{w} \in L(S_F)$.

Setting $t_0 = m_0 - 2(N-1)$ and repeating the same construction for $m > m_0$ completes the proof. ∎

3.2 PROPERTIES OF THE UNIFORM MEASURE. The flows considered until now are not necessarily bijective. The canonical way to make them so is to consider their natural extensions (see for instance [CFS]). The **natural extension** of a flow (X, T) is the flow $(\tilde{X}, \tilde{T})$, where $\tilde{X} = \{\tilde{x} = (x^{(0)}, x^{(1)}, \ldots) \in X^{\mathbb{N}}$: for all $i \geq$

$0,\ T(x^{(i+1)}) = x^{(i)}\}$ and $\tilde{T}((x^{(0)}, x^{(1)}, \ldots)) = (T(x^{(0)}), x^{(0)}, x^{(1)}, \ldots)$. $\tilde{T}$ is obviously an automorphism of $\tilde{X}$. The flow (X, T) is a factor of $(\tilde{X}, \tilde{T})$ for the projection with respect to the first coordinate. Let μ be a T-invariant measure, and $\mathcal{B}$ be the Borel sigma-algebra on X. We define $\tilde{\mu}$ over the sigma-algebra $\tilde{\mathcal{B}}$, generated by the family $C^{(i)} = \{\tilde{x} \in \tilde{X}: x^{(i)} \in C\}$, $C \in \mathcal{B}$, by: $\tilde{\mu}(C^{(i)}) = \mu(C)$.

The natural extension of a one-sided subshift S is the two-sided subshift $\tilde{S}$ such that $L(S) = L(\tilde{S})$. Thus, the natural extension $(\tilde{A}^{\mathbb{N}}, \tilde{F})$ of a positively expansive CA is conjugate to $(\tilde{S}_F, \sigma)$. Recall λ denotes the uniform measure on $A^{\mathbb{N}}$ and $\lambda_F = \pi_F(\lambda)$.

THEOREM 3.9: *Let F: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ be positively expansive. Then* $h_\lambda(A^{\mathbb{N}}, F) = h_{\lambda_F}(S_F) = \log k_1 = h_{\mathrm{top}}(A^{\mathbb{N}}, F)$.

Proof: By conjugacy we only have to consider the case where F is of type (E1). Recall that S_F is of order N. Since $(A^{\mathbb{N}}, F, \lambda)$ is conjugate to (S_F, σ, λ_F), by equality (2.1) we have to compute

$$h_{\lambda_F}(S_F) = \lim_{n\to\infty} -\frac{1}{n} \sum_{w \in L_n(S_F)} \lambda_F([w]_0) \log \lambda_F([w]_0).$$

Fix $n \geq N - 1$. For $w = w_0 \cdots w_{n-1} \in L_n(S_F)$, let us show first that $\lambda_F([w]_0)$ depends only on n and the last $N - 1$ letters of w. By definition,

$$\begin{aligned}\lambda_F([w]_0) &= \lambda(\pi_F^{-1}([w]_0)) \\ &= \frac{1}{\mathrm{card}(A)^n}\ \mathrm{card}(\{w' \in A^n: f^i(w')(0) = w_i,\ i = 0, \ldots, n-1\}).\end{aligned}$$

Denote by $u = u_0 \cdots u_{N-2}$ the suffix of length $N - 1$ of w and by $k^*(u)$ the cardinality of the set

$$S(u) = \{v \in A^{N-1}:\ f^i(v)(0) = u_i,\ i = 0, \ldots, N-2\}.$$

By Lemmas 3.4 and 3.5, for each $v \in S(u)$ there exist k_2^{n-N+1} words $w' \in A^n$ such that $f^i(w')(0) = w_i$, $i = 0, \ldots, n-1$, and $f^{n-N+1}(w')(0, N-2) = v$. Therefore

$$\mathrm{card}(\{w' \in A^n: f^i(w')(0) = w_i,\ i = 0, \ldots, n-1\}) = k^*(u)\ k_2^{n-N+1},$$

and

$$\lambda_F([w]_0) = \frac{1}{\mathrm{card}(A)^n}\ k^*(u)\ k_2^{n-N+1}, \tag{3.4}$$

which by Proposition 3.6 can be rewritten

$$\lambda_F([w]_0) = \frac{1}{k_1^n}\, C\, k^*(u),$$

where C does not depend on n and w. Noticing that the last equality implies

$$k_1^n = \sum_{w \in L_n(S_F)} C\, k^*(u),$$

by a simple computation one obtains $h_{\lambda_F}(S_F) = \log k_1$. ∎

Consequently $\tilde{\lambda}$ and $\tilde{\lambda}_F$ are measures of maximal entropy for their respective flows. But by Theorem 3.8 $\tilde{S}_F$ is a mixing SFT, which by Parry's result ([P1] or theorem (19.14) in [DGS]) bears a unique measure of maximal entropy $\mu_{\max}$ called the Parry measure; this measure must coincide with $\tilde{\lambda}_F$.

The following corollary follows directly from the equality of $\tilde{\lambda}_F$ with the Parry measure.

COROLLARY 3.10: *Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be positively expansive. Then the natural extension $(\tilde{A}^{\mathbb{N}}, \tilde{F}, \tilde{\lambda}_F)$ is measure-theoretically isomorphic to the full shift $\{0, \ldots, k_1 - 1\}^{\mathbb{Z}}$ endowed with the uniform measure.*

Proof: Since $\tilde{\lambda}_F$ is the Parry measure of a mixing SFT, by [DGS] (17.15, corollary 4), $(\tilde{A}^{\mathbb{N}}, \tilde{F}, \tilde{\lambda}_F)$ is a Bernoulli system of entropy $\log k_1$. Now, by the Ornstein theorem (see [DGS] (12.10)) it is isomorphic to the full k_1-shift endowed with the uniform measure. ∎

Since $\tilde{\lambda}_F$ is a Parry measure, it is Markov, and therefore has some probabilistic properties of asymptotic independance that may be exploited for the study of λ.

Some additional combinatorial properties, which will be used farther, can be deduced from the equality $\lambda_F = \mu_{\max}$. Let s^*: $A^* \to A^{N-1}$ be the map that gives the suffix of length $N-1$ of a word; for each automaton of type (E1) let k^*: $A^{N-1} \to \mathbb{N}$ be the map associating to each $u \in A^{N-1}$ the cardinality of the set $S(u) = \{v \in A^{N-1}\colon f^i(v)(0) = u_i,\ i = 0, \ldots, N-2\}$. By subsection 2.1, we know that for a positively expansive CA, F, $\tilde{S}_F$ is conjugate to a Markov system $\tilde{S}'_F \subseteq ((A^r)^{N-1})^{\mathbb{Z}}$; call M_F its transition matrix.

COROLLARY 3.11: *Let F be a CA of type* (E1). *Then $(k^*(w)\ ;\ w \in L_{N-1}(S_F))$ is a right eigenvector of M_F for the eigenvalue k_1.*

Proof: First, by Proposition 3.6 and equality (3.4), for each word $w \in L_{n+N-1}(S_F)$, $n \geq 0$ one has

$$\tilde{\lambda}_F([w]_0) = \frac{k^*(s^*(w))}{\text{card}(A)^{N-1}} \ \frac{1}{k_1^n}. \tag{3.5}$$

The isomorphic image of $\tilde{\lambda}$ on $\tilde{S}'_F$ has maximal entropy; it is defined for all $w' = w'_0 \cdots w'_{n-1} \in L_n(\tilde{S}'_F)$, $i \in \mathbb{Z}$, by:

$$\tilde{\lambda}'([w']_i) = P_F(w'_0, w'_1)P_F(w'_1, w'_2) \ \cdots \ P_F(w'_{n-2}, w'_{n-1}) \ \lambda_F(w'_{n-1}),$$

where

$$\lambda_F(w) = \frac{k^*(w)}{\text{card}(A)^{N-1}} \quad \text{and} \quad P_F(w, \bar{w}) = \frac{1}{k_1} \ M_F(w, \bar{w}),$$

for all $w, \bar{w} \in L_{N-1}(S_F)$.

On the other hand, since the sum of the columns of M_F is k_1, the associated Parry measure, denoted by μ_P, satisfies the following property: for all $i \in \mathbb{Z}$, $w \in L_{N-1}(S_F)$, $\mu_P([w]_i) = v(w)$, where $(v(w);\ w \in L_N(S_F))$ is a normalized right eigenvector of M_F for the eigenvalue k_1 ([DGS]). Identifying μ_P and $\tilde{\lambda}'_F$ one gets

$$\sum_{\{a \in A:\ wa \in L_N(S_F)\}} k^*(s^*(wa)) = k_1 \ k^*(w), \tag{3.6}$$

in other words $(k^*(w)\ ;\ w \in L_{N-1}(S_F))$ is an eigenvector of M_F for k_1. ∎

In particular, if $N = 2$, Corollary 3.11 implies that for all $a \in A$, $\text{card}\{b \in A : ab \in L_2(S_F)\} = k_1$.

3.3 Further combinatorial properties. We just proved that the natural extensions of positively expansive one-sided cellular automata, endowed with the uniform measure, are metrically isomorphic to Bernoulli systems. We now go on with the combinatorial study, which allows us to prove that the canonical factors of positively expansive one-sided CA are shift equivalent to full shifts, but also leads to some arithmetic obstructions to expansiveness which we use for obtaining a necessary and sufficient condition in Proposition 4.2.

The following result is a quantitative version of Theorem 3.8. Recall that M_F denotes the incidence matrix of the Markov system $\tilde{S}'_F \subseteq (A^{N-1})^{\mathbb{Z}}$ conjugate to $\tilde{S}_F$; the notations k^* and s^* have been introduced before Corollary 3.11.

PROPOSITION 3.12: *Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be positively expansive. There exists $t_0 > N$ such that for all $t \geq t_0$ and $w, w' \in L_{N-1}(S_F)$,*

$$M_F^t(w, w') = c_t(w) = \frac{k_1^t \; k^*(w)}{\mathrm{card}(A)^{N-1}}.$$

Proof: Without loss of generality assume that F is of type (E1). By Theorem 3.8 and the existence of the transposed flow $\bar{F}$ (Lemma 2.4), there exists $t_0 > N$ such that $M_F^t > 0$ for all $t \geq t_0$, and that for all $x \in A^{\mathbb{N}}$ the words

$$F^0(x)(N-1) \cdots F^{N-2}(x)(N-1) \quad \text{and} \quad x(0, N-2)$$

are uniquely determined by $F^i(x)(0)$, $i = 0, \ldots, t_0 + N - 2$.

Fix $w, w' \in L_{N-1}(S_F)$ and $u = u_0 \cdots u_{N-2} \in A^{N-1}$. We shall prove that

$$M_F^{t_0}(w, w') = \sum_{v \in C_{t_0}(u,w)} k^*(s^*(v)), \tag{3.7}$$

where

$$C_{t_0}(u, w) = \{v \in L_{t_0+N-2}(S_F) \colon u_0 v \xrightarrow{\bar{f}^{N-1}} w, \; \bar{f}^i(u_0 v)(0) = u_i, \; i = 0, \ldots, N-2\}.$$

It is straightforward that $C_{t_0}(u, w) \neq \emptyset$ for all $u \in A^{N-1}$, $w \in L_{N-1}(S_F)$.

Let $v = v_0 \cdots v_{t_0+N-3} \in C_{t_0}(u, w)$ and fix $\tilde{u} \in A^{N-1}$ such that $f^i(\tilde{u})(0) = s^*(v)_i = v_{t_0-1+i}$, $i = 0, \ldots, N-2$. Since

$$M_F^{t_0}(w, w') = \mathrm{card}(\{\tilde{w} \in A^{t_0-N+1} \colon w\tilde{w}w' \in L_{t_0+N-1}(S_F)\}),$$

to prove (3.7) we only have to show, on the one hand, that once v is fixed, the choice of $\tilde{u}$ determines $\tilde{w}$ and, conversely, on the other hand, that u and $w\tilde{w}w'$ determine v. For a quicker understanding see Figure 3.1 below.

By the same arguments as in the proof of Theorem 3.8 one shows that to any $\tilde{z} \in A^{\mathbb{N}}$ such that $F^0(\tilde{z})(0) \cdots F^{N-2}(\tilde{z})(0) = w'$ there corresponds a unique $z \in A^{\mathbb{N}}$ with (i) $F^0(z)(0) \cdots F^{N-2}(z)(0) = w$, (ii) $F^{t_0}(uz) = \tilde{u}\tilde{z}$, and (iii) $F^i(uz)(0) = v_{i-1}$, $i = 1, \ldots, t_0 + N - 2$. Recall that z only depends on u, $\tilde{u}$, $\tilde{z}$, and v. Putting

$$\tilde{w} = F^{N-1}(z)(0) \cdots F^{t_0-1}(z)(0) \in A^{t_0-N+1},$$

then $w\tilde{w}w' \in L(S_F)$; we say that $\tilde{w}$ links up w with w' in the context of $(u, v, \tilde{u})$.

Once v and $\tilde{u}$ are fixed, $\tilde{w}$ is unique. In fact, if $\tilde{w}' \in A^{t_0-N+1}$ links up w with w' in the context of $(u, v, \tilde{u})$, then for all $\tilde{z}' \in A^{\mathbb{N}}$ such that $F^0(\tilde{z}')(0)\cdots F^{N-2}(\tilde{z}')(0) = w'$ there exists a unique $z' \in A^{\mathbb{N}}$ such that

$$F^0(z')(0)\cdots F^{t_0+N-2}(z')(0) = w\tilde{w}'w' \quad \text{and} \quad F^{t_0}(z') = \tilde{z}'.$$

Choose $\tilde{z}' = \tilde{z}$. Therefore $\tilde{u}\tilde{z}' = \tilde{u}\tilde{z}$; then, since v has been fixed, and by using the expansiveness of F, we deduce that $uz' = uz$ and $z' = z$. Finally, by applying F several times one concludes that $\tilde{w}' = \tilde{w}$ (recall that the block B in Figure 3.1 is determined by u and w, and the block C by u, w and $\tilde{w}$). Hence for each quintuple $(w, w', u, v, \tilde{u})$ there is a unique word $\tilde{w} \in A^{t_0-N+1}$ which links up w with w' in the context of $(u, v, \tilde{u})$.

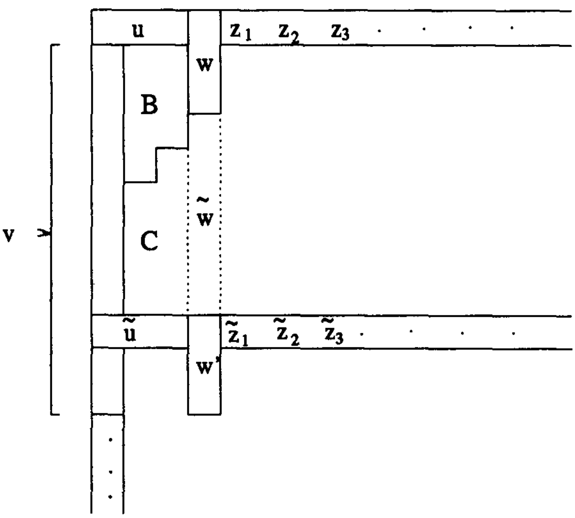

Figure 3.1

On the other hand, if $\tilde{u}_1$ and $\tilde{u}_2$ are different words such that $f^i(\tilde{u}_j)(0) = s^*(v)_i$, $i = 0, \ldots, N-2$, $j = 1, 2$, then the associated words $\tilde{w}_j$ are necessarily different: the proof of this fact, closely related to the proof of the unicity of $\tilde{w}$, is left to the reader. Thus, once $u \in A^{N-1}$ has been fixed, to each $v \in C_{t_0}(u, w)$ we can associate exactly $k^*(s^*(v))$ words in $\{\tilde{w} \in A^{t_0-N+1}\colon w\tilde{w}w' \in L_{t_0+N-1}(S_F)\}$.

But every word of $\{\tilde{w} \in A^{t_0-N+1}\colon w\tilde{w}w' \in L_{t_0+N-1}(S_F)\}$ is associated with a unique $v \in C_{t_0}(u, w)$ (it is uniquely determined by a suitable number of applications of f when u and $w\tilde{w}w'$ have been fixed); we have thus obtained that

$$M_F^{t_0}(w, w') = \sum_{v \in C_{t_0}(u,w)} k^*(s^*(v));$$

consequently $M_F^{t_0}(w,w') = c_{t_0}(w)$ only depends on w.

To finish, by Corollary 3.11 we know that $(k^*(w);\ w \in L_{N-1}(S_F))$ is a right eigenvector of M_F for the eigenvalue k_1; then

$$\sum_{w' \in L_{N-1}(S_F)} M_F^{t_0}(w,w')\ k^*(w') = c_{t_0}(w)\ \operatorname{card}(A)^{N-1} = k_1^{t_0} k^*(w),$$

so we deduce the result for $t = t_0$, and afterwards for $t > t_0$. ∎

Corollary 3.13: *Let F: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ be positively expansive, and let k_1, k_2 be the two integers such that $h_{\text{top}}(F) = \log k_1$ and $\operatorname{card}(A)^r = k_1\ k_2$. Then k_2 divides an integer power of k_1.*

Proof: By recoding we can assume that F is of type (E1). Let $w, w' \in L_{N-1}(S_F)$ and $L \in \mathbb{N}$. If we follow the construction in Proposition 3.12 but with $u \in A^{N-1+L}$ we obtain that for all $t \geq t_0(L)$

$$M_F^t(w,w') = \operatorname{card}(A)^L \sum_{v \in \bar{C}_t(u,w)} k^*(s^*(v)),$$

where $\bar{C}_t(u,w)$ is defined in the same way as $C_t(u,w)$ in the former proposition. On the other hand, by Proposition 3.12,

$$M_F^t(w,w') = \frac{k_1^{t-N+1} k^*(w)}{k_2^{N-1}},$$

hence

$$\frac{k_1^{t-N+1}\ k^*(w)}{k_2^{N-1}} = \operatorname{card}(A)^L K(t,w,u),$$

where $K(t,w,u)$ is a positive integer. Since $k^*(w)$ and k_2^{N-1} contain a fixed number of powers of prime numbers and they do not depend on t, then with L big enough we conclude that k_1 has all the prime divisors of $\operatorname{card}(A)$. By definition k_2 must divide some integer power of k_1. ∎

Recall that two non-negative integer square matrices A and B are shift equivalent if there exist non-negative integer matrices R, S and a positive integer l such that (i) $A \cdot R = R \cdot B$, (ii) $S \cdot A = B \cdot S$, (iii) $A^l = R \cdot S$ and (iv) $B^l = S \cdot R$. Shift equivalence corresponds to the eventual conjugacy of the SFT X_A and X_B defined by the matrices A and B respectively, that is, (X_A, σ^m) and (X_B, σ^m) are conjugate for sufficiently large m (for more details about shift equivalence see [BMT]).

COROLLARY 3.14: *Let $F: A^{\mathbb{N}} \to A^{\mathbb{N}}$ be positively expansive. Then M_F is shift equivalent to $[k_1]$, where $h_{\text{top}}(F) = \log k_1$. In particular, $(\tilde{S}_F, \sigma)$ is eventually conjugate to $(\{0, \dots, k_1 - 1\}^{\mathbb{Z}}, \sigma)$.*

Proof: Let $t_0 > N$ be the constant given in Proposition 3.12 such that $M_F^{t_0}(w, w') = c_{t_0}(w)$ for any w, $w' \in L_{N-1}(S_F)$. From Corollary 3.11 and Proposition 3.6 one gets that $l = (1, \dots, 1)$ and $r = (c_{t_0}(w);\ w \in L_{N-1}(S_F))$ are respectively left and right eigenvectors of M_F associated to k_1. Thus, $M_F \cdot r = r \cdot k_1$, $l \cdot M_F = k_1 \cdot l$, $r \cdot l = M_F^{t_0}$ and $l \cdot r = k_1^{t_0}$. These last equalities prove the corollary. ∎

4. The multiplication by certain integers in base p

The examples in this section are mainly here for illustration. In order to show that the cellular automaton of multiplication by k, endowed with the uniform measure, is a Bernoulli system, Corollary 3.10 is not needed: this results from the fact that $\{0, \dots, p-1\}^{\mathbb{N}}$, endowed with the automaton, is an almost conjugate representation of the multiplication by k on the torus. Our purpose is to describe a family of positively expansive CA, derived from Arithmetics, that are neither toggle automata, nor permutative in a weak sense like Example 2.

Consider a cellular automaton F of type (E1) and suppose that S_F is a Markov system. We proved in the last section that there are constants $k_1, k_2 \in \mathbb{N}$, where k_2 divides an integer power of k_1, such that $\text{card}(A) = k_1\ k_2$. Furthermore, for all $a \in A$, one has $k_1 = \text{card}(\{a' \in A\colon a \xrightarrow{f} a'\}) = \text{card}(\{a' \in A\colon a' \xrightarrow{f} a\})$, and $k_2 = \text{card}(\{b \in A\colon f(ab) = a'\})$ for all $a' \in A$ such that $a \xrightarrow{f} a'$. We construct positively expansive cellular automata for each pair of integers (k_1, k_2) satisfying the former conditions. For $p \in \mathbb{N}$ put $A_p = \{0, \dots, p-1\}$.

First fix a pair $(k_1, k_2) \in \mathbb{N}^2$ and put $p = k_1\ k_2$ (for the moment we do not suppose that k_2 divides an integer power of k_1). Each configuration of $A_p^{\mathbb{N}}$ is the expansion in base p of a real number in the interval $[0, 1]$: it is proved in [BHM] that for all $k \in \mathbb{N}$ such that k divides an integer power of p there exists a CA on the alphabet A_p, which represents the multiplication by k in base p. This follows from the fact that, under the last condition, the algorithm of multiplication by an integer in base p only depends on a finite number of carries.

In particular, consider the case $k = k_1$. Each integer $a \in A_p$ can be written $a = f_a k_2 + l_a$ where $f_a \in A_{k_1}$ and $l_a \in A_{k_2}$, or also $a = \bar{f}_a k_1 + \bar{l}_a$ with $\bar{f}_a \in A_{k_2}$ and $\bar{l}_a \in A_{k_1}$. It is not difficult to check that the set of carries obtained in the multiplication by k_1 in base p is equal to A_{k_1}. Let $a \in A_p$: independently of the value of $r \in A_{k_1}$, the carry obtained when we multiply a by k_1 and add r is always f_a; in fact, $k_1 a + r = k_1 k_2 f_a + l_a k_1 + r = f_a p + (l_a k_1 + r)$, where $l_a k_1 + r \leq p - 1$. The onto cellular automaton F_{k_1,k_2} representing the multiplication by k_1 in base p is then defined by

$$F_{k_1,k_2}(x)_i = k_1\ l_{x_i}\ +\ f_{x_{i+1}}, \quad x \in A_p^{\mathbb{N}}, \quad i \in \mathbb{N}. \tag{4.1}$$

It is clear this formula defines a cellular automaton with radius 1. The corresponding map is always right resolving: recall that a CA F: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ is right resolving if for any $a, b, \beta \in A^r$ such that $a \xrightarrow{f} b$ there is a unique $\alpha(\beta) \in A^r$ such that $a\alpha(\beta) \xrightarrow{f} b\beta$. Let us prove the former statement: choose $a, a' \in A_p$ such that $a \xrightarrow{f_{k_1,k_2}} a'$. From the definition of F_{k_1,k_2} one gets $a' = l_a k_1 + \bar{l}_{a'}$, and if $b \in A_p$ satisfies $f_{k_1,k_2}(ab) = a'$ then $b = \bar{l}_{a'} k_2 + l_b$. In particular, the value of a' does not depend on f_a. Consequently, for all $b' \in A_p$ there is a unique $b \in A_p$ such that $ab \xrightarrow{f_{k_1,k_2}} a'b'$: recall that $b = \bar{l}_{a'} k_2 + \bar{f}_{b'}$.

Proposition 4.1: *Let F: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ be a right resolving cellular automaton. Then S_F is a Markov system.*

Proof: We have to prove that $a_1 \cdots a_n \in L_n(S_F)$ whenever $a_i \xrightarrow{f} a_{i+1}$ for $i = 1, \ldots, n-1$. Let us use induction on n. Let $b_{n-1} \in A^r$ be such that $f(a_{n-1}, b_{n-1}) = a_n$ (it exists by hypothesis). Since $a_{n-2} \xrightarrow{f} a_{n-1}$ and F is right resolving there is a unique $b_{n-2} \in A^r$ such that $f(a_{n-2}, b_{n-2}) = a_{n-1}$ and $b_{n-2} \xrightarrow{f} b_{n-1}$. Using the same argument inductively we find a word $b_1 \cdots b_{n-1} \in A^{r(n-1)}$ such that $b_i \xrightarrow{f} b_{i+1}$ for $i = 1, \ldots, n-2$, and $f(a_i, b_i) = a_{i+1}$ for $i = 1, \ldots, n-1$. Now use the induction hypothesis to conclude that $a_1 \cdots a_n \in L_n(S_F)$. ∎

In particular the last proposition implies that $S_{F_{k_1,k_2}}$ is a Markov system.

We proved in Corollary 3.13 that expansiveness implies $k_1^{\ell} = k_2\ k_3$ for some integers $\ell \in \mathbb{N}$, $k_3 \in \mathbb{N}$. In particular, if k_1 and k_2 are relatively prime F_{k_1,k_2} is not expansive.

Proposition 4.2: *The cellular automaton F_{k_1,k_2} is positively expansive if and only if k_2 divides a positive power of k_1.*

Proof: The necessary condition follows directly from Corollary 3.13. Conversely, suppose that k_2 divides a positive power of k_1; then $k_1 = p_1^{s_1} \cdots p_t^{s_t}$ and $k_2 = p_1^{\bar{s}_1} \cdots p_t^{\bar{s}_t}$, where $s_i > 0, \bar{s}_i \geq 0$, $i = 1, \ldots, t$, and $p_1, \ldots, p_t$ are prime numbers.

Consider the sequence $(m_i)_{i\in\mathbb{N}}$ defined inductively by: (i) $m_0 = k_2$, (ii) for all $i \geq 1$, $m_i = \text{MCM}(m_{i-1}, k_1) : k_1$. One proves by induction that the prime factors of m_i are among those of k_1, so $m_i < m_{i-1}$ whenever m_{i-1} does not divide k_1. Consequently, there is a smallest integer $T \in \mathbb{N}$ such that m_T divides k_1 (and for $i > T$, $m_i = 1$). Also, for all i, m_i divides m_{i-1}. Denote by $\bar{m}_i$ and n_i the integers such that $m_i\bar{m}_i = p$ and $n_i m_i = k_2$. To each m_i, $i = 0, \ldots, T$, associate the partition $\mathcal{P}_i = \{C_{0,i}, \ldots, C_{\bar{m}_i-1,i}\}$ of A_p, where $C_{j,i} = \{jm_i, \ldots, (j+1)m_i - 1\}$, $j = 0, \ldots, \bar{m}_i - 1$. Notice that $\mathcal{P}_{i+1}$ is a refinement of $\mathcal{P}_i$. Fix $i \in \{0, \ldots, T\}$. We shall prove the following property: let us consider $a, a', b, b' \in A_p$; if on the one hand a and b, and on the other hand a' and b', belong to the same atom of the partition $\mathcal{P}_i$, and $a \overset{f_{k_1,k_2}}{\longrightarrow} a'$, $b \overset{f_{k_1,k_2}}{\longrightarrow} b'$, then a and b belong to the same atom of the partition $\mathcal{P}_{i+1}$.

Let C be an atom of $\mathcal{P}_i$. There exist $f \in A_{k_1}$, $f_i \in A_{n_i}$ such that each $a \in C$ can be written $a = fk_2 + f_i m_i + f_{a,i+1} m_{i+1} + l_{a,i+1}$, where $f_{a,i+1} \in A_{m_i/m_{i+1}}$ and $l_{a,i+1} \in A_{m_{i+1}}$. Consider $a, b \in C$ such that $f_{a,i+1} < f_{b,i+1}$; in other words they are not in the same atom of the partition $\mathcal{P}_{i+1}$; denote by C_a and C_b the atoms of $\mathcal{P}_{i+1}$ containing a and b. It follows directly from the definition of F_{k_1,k_2} that the set of $a' \in A_p$ such that $a'' \overset{f_{k_1,k_2}}{\longrightarrow} a'$ for some $a'' \in C_a$ is

$$S_a = \{k_1\ f_i\ m_i + k_1\ f_{a,i+1}\ m_{i+1} + k_1\ l + r \colon l \in A_{m_{i+1}},\ r \in A_{k_1}\}.$$

Analogously,

$$S_b = \{k_1\ f_i\ m_i + k_1\ f_{b,i+1}\ m_{i+1} + k_1\ l + r : l \in A_{m_{i+1}},\ r \in A_{k_1}\}.$$

A simple computation shows that

$$S_a = [j_a m_i, j_a m_i + \text{MCM}(m_i, k_1) - 1] \text{ and } S_b = [j_b m_i, j_b m_i + \text{MCM}(m_i, k_1) - 1],$$

where

$$j_a = k_1\ f_i + f_{a,i+1}\ \frac{\text{MCM}(m_i, k_1)}{m_i} \quad \text{and} \quad j_b = k_1\ f_i + f_{b,i+1}\ \frac{\text{MCM}(m_i, k_1)}{m_i}$$

($[i, j]$ denotes the interval of integers from i to j). Then S_a and S_b are unions of atoms of $\mathcal{P}_i$. Furthermore, since $f_{a,i+1} < f_{b,i+1}$, then S_a and S_b are disjoint. We proved that a and b do not have a successor in the same atom of $\mathcal{P}_i$ whenever $a, b \in C$ and $C_a \neq C_b$, which implies the desired property.

Suppose now that $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}} \in S_{F_{k_1,k_2}}$ are such that $f_{a_i} = f_{b_i}$ for all $i \in \mathbb{N}$; in other words, for each i, a_i and b_i are in the same atom of $\mathcal{P}_0$. Applying the last result T times, we deduce that for all $i \in \mathbb{N}$, a_i and b_i belong to the same atom of the partition $\mathcal{P}_{T+1}$, which is the discrete partition, so $(a_i)_{i\in\mathbb{N}} = (b_i)_{i\in\mathbb{N}}$. Since the radius of F_{k_1,k_2} is 1, this establishes the expansiveness of the CA. ∎

COROLLARY 4.3: *Let k_1 and k_2 be natural integers such that k_2 divides a power of k_1. Then $S_{F_{k_1,k_2}}$ is conjugate to the one-sided full shift over k_1 letters.*

Proof: It follows from the definition of $S_{F_{k_1,k_2}}$ and from Proposition 4.1 that there is a partition $\mathcal{P} = \{A_1, \ldots, A_{k_1}\}$ of A such that $\mathrm{card}(A_i) = k_2$ for $i = 1, \ldots, k_1$ and for any b, $b' \in A_i$, $a \in A$ $f_{k_1,k_2}(ab) = f_{k_1,k_2}(ab')$. Define the map $\bar{\pi}_{F_{k_1,k_2}} : S_{F_{k_1,k_2}} \to \{1, \ldots, k_1\}^{\mathbb{N}}$ by $\bar{\pi}_{F_{k_1,k_2}}(x)_j = i$ if and only if $x_j \in A_i$ $(j \in \mathbb{N})$. By Lemma 3.2, $\bar{\pi}_{F_{k_1,k_2}}$ is onto. To prove it is 1-to-1 consider $x^1, x^2 \in S_{F_{k_1,k_2}}$ such that $\bar{\pi}_{F_{k_1,k_2}}(x^1) = \bar{\pi}_{F_{k_1,k_2}}(x^2)$. Then for all $i \in \mathbb{N}$, x_i^1 and x_i^2 belong to the same atom of the partition $\mathcal{P}$, that is, for all $a \in A$, $f_{k_1,k_2}(ax_i^1) = f_{k_1,k_2}(ax_i^2)$. Thus there exists a point $x \in S_{F_{k_1,k_2}}$ such that $\bar{F}_{k_1,k_2}(x) = x^1 = x^2$. The result follows from the expansiveness of F_{k_1,k_2}. ∎

Proposition 4.2 and Corollary 4.3 state a necessary and sufficient condition for the cellular automaton representing the multiplication by k in base p to be topologically conjugate to a full shift over k symbols (which is, by the way, a more convenient symbolic representation of this multiplication !).

5. Comments and questions

1. We mention that expansiveness plays an important part in the classification of CA by Gilman [G] as well as in the recent topological classification of Kůrka [Ků].

2. Positively expansive one-sided CA maps form a strict subset of the family of right-closing maps; the latter have been widely used in Symbolic Dynamics. It is natural to ask whether some of the results in this article can be extended to this family: for instance, is the subshift S_F always of finite type in this case? Of course right-closing CA are not generally mixing (the identity map is not) and S_F is not always shift equivalent to a full shift (multiplication by 2 in base 6).

3. Recently M. Boyle, D. and U. Fiebig found an expansive CA the canonical factor of which is not conjugate to a one-sided full shift (private communication).

4. In [Cou] M. Courbage proved that a particular invertible CA, with the property of (non-positive) expansivity, possesses most of the properties we obtain here. Can this be proved for all (non-positively) expansive automorphisms of the full shift ?

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